

NOTES ON PHASE SPACE QUANTIZATION

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ABSTRACT. We consider questions related to a quantization scheme in which a classical variable $f : \Omega \rightarrow \mathbb{R}$ on a phase space Ω is associated with a semispectral measure E^f , such that the moment operators of E^f are required to be of the form $\Gamma(f^k)$, with Γ a suitable mapping from the set of classical variables to the set of (not necessarily bounded) operators in some Hilbert space. In particular, we investigate the situation where the map Γ is implemented by the operator integral with respect to some fixed positive operator measure. The phase space Ω is first taken to be an abstract measurable space, then a locally compact unimodular group, and finally \mathbb{R}^2 , where we determine explicitly the relevant operators $\Gamma(f^k)$ for certain variables f , in the case where the quantization map Γ is implemented by a translation covariant positive operator measure. In addition, we consider the question under what conditions a positive operator measure is projection valued.

1. INTRODUCTION

Quantization can be any procedure which associates a quantum mechanical observable to a given classical dynamical variable. The traditional way to realize a quantization is to assign to each classical variable a Hermitean (symmetric, or even essentially selfadjoint) operator which should describe the quantum observable. In the modern view of a quantum observable as a semispectral measure, this kind of quantization is no longer sufficient. The question is how to modify the traditional scheme in order to fit it into the context of modern quantum mechanics.

Classical variables can be represented by real valued measurable functions defined on some measurable space (Ω, \mathcal{A}) , which is the phase space of the classical system. The phase space can be taken to be e.g. \mathbb{R}^{2n} , in which case the variables are Borel functions. In the conventional approach to quantization, we would have a map Γ from the set of real measurable functions to the set of all linear (not necessarily bounded) operators in \mathcal{H} , and $\Gamma(f)$ would be the observable corresponding to the classical variable f (see e.g [9, 15, 28, 32, 33]). We would like to modify this scheme so that we could assign a semispectral measure to the function f , instead of an operator. To do this, we use all the operators $\Gamma(f^k)$, $k \in \mathbb{N}$, instead of just the first one of them, and then consider the moment problem of finding the unique semispectral measure $E^f : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ with the property that $\int x^k dE^f = \Gamma(f^k)$ for all $k \in \mathbb{N}$. Here $\int x^k dE^f$ is the operator integral $L(x^k, E^f)$ of the function $x \mapsto x^k$ with respect to E^f ; see the next Section for its definition. So if Γ and f are such that there exists a unique solution E^f of the described moment problem, then the collection of the operators $\{\Gamma(f^k) \mid k \in \mathbb{N}\}$ is eligible to represent a quantization E^f of f .

We mention that the approach described above is used, for instance, in [14] in the description of a quantum measurement. In a typical measurement situation, where one aims to measure a (traditional) quantum observable represented by a selfadjoint operator, one is actually measuring a noisy or unsharp version of that observable. The noisy version, represented by a semispectral measure, may agree with the observable intended to be measured, on the statistical level of expectations. The unsharpness of the measurement is reflected in the fact that the

dispersion of the measurement statistics is actually greater than what would be obtained in the noiseless case. The observable, as represented by a semispectral measure, cannot be described using a single operator. Accordingly, in the approach of [14], each moment of the measurement outcome distribution is considered to be an average of certain operator, called "operational observable", and the collection of these operators then represents the measured observable. We want to point out that these "operational observables" are nothing else but the moment operators of the semispectral measure representing the observable. Note that these moments need not determine the semispectral measure uniquely. In certain special cases, however, they do [12, 13], and sometimes even the first moment is enough [8, 25].

One way to obtain a quantization map Γ is to use the operator integral with respect to some given positive operator measure E , i.e. define Γ to be the map $f \mapsto L(f, E)$. In this case, if we define E^f on the Borel sets of the real line by $B \mapsto E(f^{-1}(B))$, we have simply $L(x^k, E^f) = L(f^k, E)$, as is easily seen by using the definition of the next Section and the usual change of variables in the integral with respect to a complex measure. So then E^f is a solution to the moment problem described above, which leaves us with the uniqueness question. Note that if we choose E to be a spectral measure, we end up getting only spectral measures as the quantized observables, with all the operators $L(f^k, E)$ mutually commuting. We also remark that, in any case, all the observables obtained via this type of quantization are *functionally coexistent* [26], so that they can be measured together in the sense of Ludwig [29, D.3.1, p. 153].

The structure of this note is as follows. In the Section 2 we give some results on the theory of operator integrals and give a simple characterization of the quantization maps which can be represented by an operator integral with respect to a positive operator measure. In Section 3, we consider the operator integral with respect to covariant positive operator measures on a locally compact topological group. Section 4 is devoted to the quantizations obtained by using the covariant phase space observables in \mathbb{R}^2 , and in Section 5, we discuss the "optimal" choice for such a covariant observable, in view of quantization. In the last Section, we consider the question under what conditions a positive operator measure is a spectral one.

2. THE OPERATOR INTEGRAL

The basic tool in our quantization procedure is the operator integral. It associates a linear (not necessarily bounded) operator in the Hilbert space \mathcal{H} of a quantum system to each complex measurable function on the phase space Ω . In this section we review the basic results concerning the theory of operator integrals and give some additional remarks in relation to quantization.

Let Ω be a nonempty set and \mathcal{A} a σ -algebra of subsets of Ω . Let \mathcal{H} be a complex Hilbert space and $L(\mathcal{H})$ the set of bounded operators on \mathcal{H} . Let $E : \mathcal{A} \rightarrow L(\mathcal{H})$ be a positive operator measure, i.e. a positive operator valued set function which is σ -additive with respect to weak operator topology. For each $\varphi, \psi \in \mathcal{H}$, let $E_{\psi, \varphi}$ denote the complex measure $B \mapsto \langle \psi | E(B) \varphi \rangle$. We recall that a normalized positive operator measure E is called a semispectral measure, and that a semispectral measure E is a spectral measure exactly when $E(A \cap B) = E(A)E(B)$ for all $A, B \in \mathcal{A}$. As mentioned in the Introduction, semispectral measures are also called observables.

Let $\mathcal{F}(\Omega, \mathcal{A})$, or $\mathcal{F}(\Omega)$ in brief, denote the set of all complex \mathcal{A} -measurable functions defined on Ω , and $\mathcal{O}(\mathcal{H})$ the set of all (not necessarily bounded) linear operators in \mathcal{H} . For $f \in \mathcal{F}(\Omega)$,

define

$$D(f, E) = \{\varphi \in \mathcal{H} \mid f \text{ is } E_{\psi, \varphi}\text{-integrable for each } \psi \in \mathcal{H}\},$$

and

$$\tilde{D}(f, E) = \{\varphi \in \mathcal{H} \mid |f|^2 \text{ is } E_{\varphi, \varphi}\text{-integrable}\}.$$

The following result was proved in [22]. The operator $L(f, E)$ appearing in it is called *the operator integral of f with respect to E* .

Theorem 1. (a) *The set $D(f, E)$ is a linear (not necessarily dense) subspace of \mathcal{H} , and there is a unique linear operator $L(f, E) = \int f dE$ on the domain $D(f, E)$ satisfying*

$$\langle \psi | L(f, E) \varphi \rangle = \int f dE_{\psi, \varphi}$$

for all $\psi \in \mathcal{H}$ and $\varphi \in D(f, E)$.

(b) *The set $\tilde{D}(f, E)$ is a subspace of $D(f, E)$.*

(c) *If f is real valued, $L(f, E)$ is a symmetric operator.*

(d) *While the inclusion $\tilde{D}(f, E) \subset D(f, E)$ may in general be proper, $\tilde{D}(f, E) = D(f, E)$ in the case where E is a spectral measure.*

Remark. Since the operator measure E is also strongly σ -additive, each set function $\mathcal{A} \ni B \mapsto E_\varphi := E(B)\varphi \in \mathcal{H}$, for $\varphi \in \mathcal{H}$, is an \mathcal{H} -valued vector measure. The definition of the operator integral states that $D(f, E)$ is the set of those $\varphi \in \mathcal{H}$ for which f is integrable with respect to the vector measure E_φ in the sense of [20, p. 21], and $L(f, E)\varphi = \int f dE_\varphi$ for each $\varphi \in D(f, E)$. As pointed out in [20, p. 37], this definition of integrability is equivalent to that of [10, p. 323] (for proof, see [41, Corollary 3.6]).

The vector measure approach provides an easy way to characterize the operator integral $L(f, E)$ by approximating f with bounded functions. For each $f \in \mathcal{F}(\Omega)$, and $n \in \mathbb{N}$, let \tilde{f}_n be such that $\tilde{f}_n(x) = f(x)$ if $|f(x)| \leq n$, and $\tilde{f}_n(x) = 0$ otherwise. It is a well-known fact that in the case where E is a spectral measure, we have

$$D(f, E) = \tilde{D}(f, E) = D^{f, E} := \{\varphi \in \mathcal{H} \mid \lim_{n \rightarrow \infty} L(\tilde{f}_n, E)\varphi \text{ exists}\}.$$

In addition, $L(f, E)\varphi = \lim_{n \rightarrow \infty} L(\tilde{f}_n, E)\varphi$ for all $\varphi \in D^{f, E}$ (see e.g. [11, p. 1196]). In the case of a general positive operator measure E , this need not be true. For example, take a probability measure defined on the Borel sets of \mathbb{R} , such that it has a density which is an even function, and $\int_0^\infty x d\mu(x) = \infty$. Let E be the positive operator measure $B \mapsto \mu(B)I$, and $f(x) = x$. Now f is not μ -integrable, so $D(f, E) = \{0\}$. But if $\varphi \in \mathcal{H}$, we have $\langle \psi | L(\tilde{f}_n, E)\varphi \rangle = \langle \psi | \varphi \rangle \int_{-n}^n x d\mu(x) = 0$ for all $n \in \mathbb{N}$ and $\psi \in \mathcal{H}$ because the density of μ is even, so that $D^{f, E} = \mathcal{H}$.

Thus in the general case, the existence of the limit $\lim_{n \rightarrow \infty} L(\tilde{f}_n, E)\varphi$ does not guarantee that $\varphi \in D(f, E)$. However, the following result holds.

Proposition 1. *Let $E : \mathcal{A} \rightarrow L(\mathcal{H})$ be a positive operator measure, and $f \in \mathcal{F}(\Omega)$. Then*

$$D(f, E) = \{\varphi \in \mathcal{H} \mid \lim_{n \rightarrow \infty} L(\chi_B \tilde{f}_n, E)\varphi \text{ exists for each } B \in \mathcal{A}\},$$

and $L(f, E)\varphi = \lim_{n \rightarrow \infty} L(\tilde{f}_n, E)\varphi$ for all $\varphi \in D(f, E)$.

Proof. Denote by D the right hand side of the set equality appearing in the statement. Let $\varphi \in D$, and let $B \in \mathcal{A}$, with $\eta_B = \lim_{n \rightarrow \infty} L(\chi_B \tilde{f}_n, E)\varphi$. Since each \tilde{f}_n is bounded, we can choose a sequence of \mathcal{A} -simple functions g_n , such that $|g_n(x) - \tilde{f}_n(x)| \leq \frac{1}{n}$ for each $n \in \mathbb{N}$ and $x \in \Omega$. Clearly the sequence (g_n) converges to f pointwise. Now

$$\|L(\chi_B g_n - \chi_B \tilde{f}_n, E)\varphi\| = \sup_{\|\psi\| \leq 1} \left| \int_B (g_n - \tilde{f}_n) dE_{\psi, \varphi} \right| \leq \frac{1}{n} \sup_{\|\psi\| \leq 1} |E_{\psi, \varphi}|(\Omega) \leq \frac{4}{n} \|E(\Omega)\| \|\varphi\|,$$

so that

$$\left\| \int_B g_n dE_\varphi - \eta_B \right\| \leq \frac{4}{n} \|E(\Omega)\| \|\varphi\| + \|L(\chi_B \tilde{f}_n, E)\varphi - \eta_B\|.$$

It follows that the sequence $(\int_B g_n dE_\varphi)$ of vectors converges for each $B \in \mathcal{A}$ (to η_B), so by the definition of [10, p. 323], f is integrable with respect to the vector measure E_φ , i.e. $\varphi \in D(f, E)$ (see the Remark following Theorem 1). Conversely, let $\varphi \in D(f, E)$. Since $D(f, E) = D(|f|, E)$ by definition, and for each $B \in \mathcal{A}$ the sequence $(\chi_B \tilde{f}_n)$ converges to $\chi_B f$ pointwise, with $|\chi_B \tilde{f}_n| \leq |\chi_B f|$, it follows e.g. from the dominated convergence theorem for the vector measure E_φ (see [10, p. 328]) that

$$L(\chi_B f, E)\varphi = \int \chi_B f dE_\varphi = \lim_{n \rightarrow \infty} \int \chi_B \tilde{f}_n dE_\varphi = \lim_{n \rightarrow \infty} L(\chi_B \tilde{f}_n, E)\varphi$$

for each $B \in \mathcal{A}$. Thus $\varphi \in D$, and $L(f, E)\varphi = \lim_{n \rightarrow \infty} L(\tilde{f}_n, E)\varphi$. \square

Remark.

- (a) We now have the subspace inclusions $\tilde{D}(f, E) \subset D(f, E) \subset D^{f, E}$, with each of them possibly proper. In the case where E is a spectral measure, both inclusions are equalities.
- (b) It is well known that in the case where E is a spectral measure, the domain $D(f, E)$ is dense. As seen before Proposition 1, this need not be the case in general, so a question arises what is required for E and f to make $D(f, E)$ dense.
- (c) Another difference to the spectral case is that for real valued $f \in \mathcal{F}(\Omega)$, the symmetric operator $L(f, E)$ is not necessarily selfadjoint (the adjoint need not even exist), and it seems to be difficult, in general, to determine when $L(f, E)$ might have selfadjoint extensions. In the case where f is positive and $D(f, E)$ dense, the positive symmetric operator $L(f, E)$ of course has its selfadjoint Friedrichs extension.

As noted in the Introduction, the starting point of our quantization scheme is a map Γ from real valued measurable functions to the set $\mathcal{O}(\mathcal{H})$. The following theorem characterizes those maps $\Gamma : \mathcal{F}(\Omega) \rightarrow \mathcal{O}(\mathcal{H})$ which are implemented by an operator integral. The corresponding result involving bounded functions is well known. Here the functions \tilde{f}_n are defined for each f as in Proposition 1.

Theorem 2. *A map $\Gamma : \mathcal{F}(\Omega) \rightarrow \mathcal{O}(\mathcal{H})$ coincides with the map $f \mapsto L(f, E)$ for a (clearly unique) positive operator measure E if and only if the following conditions are satisfied.*

- (i) Γ restricted to bounded functions is a positive linear map with values in $L(\mathcal{H})$;
- (ii) if (f_n) is an increasing sequence of positive \mathcal{A} -measurable functions converging pointwise to a bounded $f \in \mathcal{F}(\Omega)$, then $\sup_{n \in \mathbb{N}} \langle \varphi | \Gamma(f_n) \varphi \rangle = \langle \varphi | \Gamma(f) \varphi \rangle$ for each $\varphi \in \mathcal{H}$;
- (iii) for each $f \in \mathcal{F}(\Omega)$, the domain $D(\Gamma(f))$ of $\Gamma(f)$ consists of those vectors $\varphi \in \mathcal{H}$ for which the sequence $(\Gamma(\chi_B \tilde{f}_n)\varphi)$ of vectors converges for each $B \in \mathcal{A}$.

(iv) for each $\varphi \in D(\Gamma(f))$, the sequence $(\Gamma(\tilde{f}_n)\varphi)$ converges to $\Gamma(f)\varphi$.

Proof. Assume first that there is a positive operator measure E , such that $\Gamma(f) = L(f, E)$ for each f . The above properties follow easily: The property (i) is well known (see e.g. [2, pp. 26-28]), and (ii) follows from the monotone convergence theorem. Proposition 1 gives (iii) and (iv). Next assume that (i)-(iii) hold for a map $\Gamma : \mathcal{F}(\Omega) \rightarrow \mathcal{O}(\mathcal{H})$. By (i) the map $\mathcal{A} \ni B \mapsto E^\Gamma(B) := \Gamma(\chi_B) \in \mathcal{O}(\mathcal{H})$ is a positive operator valued additive set function. Since Γ is positive for bounded functions by (i), $\sup_{n \in \mathbb{N}}$ in condition of (ii) can be replaced by $\lim_{n \rightarrow \infty}$. This implies that each set function $B \mapsto \langle \varphi | E^\Gamma(B) \varphi \rangle$ is a positive measure, so that E^Γ is a positive operator measure. If $f \in \mathcal{F}(\Omega)$ is a bounded positive function, we have $\langle \varphi | \Gamma(f) \varphi \rangle = \langle \varphi | L(f, E^\Gamma) \varphi \rangle$ for all $\varphi \in \mathcal{H}$, as is seen by approximating f with an increasing sequence of simple functions and using linearity, (ii), and the monotone convergence theorem. Hence, if $f \in \mathcal{F}(\Omega)$ is bounded, it follows by linearity and polarization that $\Gamma(f) = L(f, E^\Gamma)$. Now let $f \in \mathcal{F}(\Omega)$ be arbitrary. It follows by (iii) and Proposition 1 that $D(\Gamma(f)) = D(f, E^\Gamma)$, and (since each \tilde{f}_n is bounded), also

$$L(f, E^\Gamma)\varphi = \lim_{n \rightarrow \infty} L(\tilde{f}_n, E^\Gamma)\varphi = \lim_{n \rightarrow \infty} \Gamma(\tilde{f}_n)\varphi = \Gamma(f)\varphi$$

for all $\varphi \in D(\Gamma(f))$ (where (iv) is used), so $L(f, E^\Gamma) = \Gamma(f)$. \square

Remark. In the quantization scheme described in the Introduction the classical variables were thought to be real valued. Obviously, the preceding Theorem holds also with $\mathcal{F}(\Omega)$ replaced by the set of real \mathcal{A} -measurable functions.

We end this section by discussing briefly a simple way of obtaining quantization maps without the use of positive operator measures. This approach is essentially the one frequently used in the conventional quantization (e.g. Weyl quantization): the operator corresponding to a given classical variable $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is obtained by integrating the variable with respect to some operator valued function defined on the phase space. The quantization of the variable then becomes a continuous distribution valued operator defined on some dense subspace of $L^2(\mathbb{R})$, which does not depend on the variable itself. In the case of Weyl quantization, for example, all the quantized operators are defined on a common domain (see e.g. [9, 36]). However, to make the situation similar to that of the operator integral map considered above, we define the quantization map in the following simple way. The proof is a direct adaptation of the proof of Theorem 1 (a) (see [22]). Note that it follows from (i) that the integrand in (ii) is \mathcal{A} -measurable. This is so because $\|\varphi\| = \sup\{|\langle \psi | \varphi \rangle| \mid \psi \in \mathcal{M}, \|\psi\| = 1\}$ for any vector $\varphi \in \mathcal{H}$, where \mathcal{M} is a fixed countable dense set in the separable Hilbert space \mathcal{H} . Hence the integral in (ii) is well defined.

Proposition 2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, \mathcal{H} a separable Hilbert space, and $\Lambda : \Omega \rightarrow L(\mathcal{H})$ a map with the following properties:

- (i) $\omega \mapsto \langle \psi | \Lambda(\omega) \varphi \rangle$ is \mathcal{A} -measurable for all $\psi, \varphi \in \mathcal{H}$;
- (ii) $\int_B \|\Lambda(\omega) \varphi\| d\mu(\omega) < \infty$ for each $\varphi \in \mathcal{H}$ and $B \in \mathcal{A}$ with $\mu(B) < \infty$.

Then for each \mathcal{A} -measurable function $f : \Omega \rightarrow \mathbb{C}$, there exists a linear operator $\Gamma_\Lambda(f)$ in \mathcal{H} , such that

$$D(\Gamma_\Lambda(f)) = \{\varphi \in \mathcal{H} \mid f \langle \psi | \Lambda(\cdot) \varphi \rangle \text{ is } \mu\text{-integrable for each } \psi \in \mathcal{H}\},$$

and

$$\langle \psi | \Gamma_\Lambda(f) \varphi \rangle = \int f(x) \langle \psi | \Lambda(\omega) \varphi \rangle d\mu(\omega) \quad \psi \in \mathcal{H}, \varphi \in D(\Gamma_\Lambda(f)).$$

Proof. It is clear that $D(\Gamma_\Lambda(f)) \subset \mathcal{H}$ is a vector subspace. Let (f_n) be a sequence of simple functions converging pointwise to f , with $|f_n| \leq |f|$ for all $n \in \mathbb{N}$. Since $|\langle \psi | \Lambda(\omega) \varphi \rangle| \leq \|\psi\| \|\Lambda(\omega) \varphi\|$ for each $\psi, \varphi \in \mathcal{H}, \omega \in \Omega$, it follows from (ii) and (i) that for each $n \in \mathbb{N}$, we have $D(\Gamma_\Lambda(f_n)) = \mathcal{H}$ and the linear functional $\psi \mapsto \int f_n(\omega) \langle \psi | \Lambda(\omega) \varphi \rangle d\mu(\omega)$ is continuous for each $\varphi \in \mathcal{H}$. Hence, for each $n \in \mathbb{N}$ and $\varphi \in \mathcal{H}$, there is $\eta_n^\varphi \in \mathcal{H}$, such that

$$\langle \psi | \eta_n^\varphi \rangle = \int f_n(\omega) \langle \psi | \Lambda(\omega) \varphi \rangle d\mu(\omega)$$

for all $\psi \in \mathcal{H}$. Now, let $\varphi \in D(\Gamma_\Lambda(f))$. Since $|f_n| \leq |f|$ for all n , the dominated convergence theorem implies that the sequence $\langle \psi | \eta_n^\varphi \rangle$ converges for each $\psi \in \mathcal{H}$ to $\int f(\omega) \langle \psi | \Lambda(\omega) \varphi \rangle d\mu(\omega)$, so by the uniform boundedness theorem and the reflexivity of \mathcal{H} , there is $\Gamma_\Lambda(f) \varphi \in \mathcal{H}$, such that

$$\langle \psi | \Gamma_\Lambda(f) \varphi \rangle = \int f(\omega) \langle \psi | \Lambda(\omega) \varphi \rangle d\mu(\omega)$$

for all $\psi \in \mathcal{H}$. Clearly, the map $D(\Gamma_\Lambda(f)) \ni \varphi \mapsto \Gamma_\Lambda(f) \varphi \in \mathcal{H}$ is linear, so the proof is complete. \square

Remark. Consider the situation where we have a locally compact unimodular topological group G , with a left Haar measure μ , and a strongly continuous projective unitary representation $U : G \rightarrow \mathcal{U}(\mathcal{H})$. Then, for any $A \in L(\mathcal{H})$, the map $\Lambda : G \rightarrow L(\mathcal{H})$, defined by $\Lambda(g) = U(g) A U(g)^*$, satisfies the conditions of the preceding Proposition, so we get the corresponding quantization map Γ_Λ . Notice that in the case where the representation is square integrable and A positive, the quantization map Γ_Λ can be represented by an operator integral if and only if A has finite trace. Namely, the case $\text{Tr}[A] = \infty$ gives $D(\Gamma_\Lambda(\chi_G)) = \{0\}$ (see e.g. [19, Lemma 2]), while the case $\text{Tr}[A] < \infty$ leads to the usual quantization map given by the operator integral with respect to a covariant positive operator measure (multiplied by some constant).

Consider the case where $G = \mathbb{R}^2$, $U(g) = U((q, p))$ are the Weyl operators, and A is the parity operator $L^2(\mathbb{R}) \ni \psi \mapsto \psi(-\cdot) \in L^2(\mathbb{R})$ (multiplied by a suitable constant). Now Γ_Λ is the Weyl quantization map. It is well known that for $f \in L^1(\mathbb{R}^2) \cup L^2(\mathbb{R}^2)$, the operator $\Gamma_\Lambda(f)$ is bounded. Moreover, if $f \in L^2(\mathbb{R}^2)$, then $\Gamma_\Lambda(f)$ is a Hilbert-Schmidt operator, and if f is a Schwartz function, then $\Gamma_\Lambda(f)$ is a trace class operator (see e.g. [36]). It is a well known fact that the Weyl quantizations of the classical position and momentum variables are the position and momentum operators Q and P , in the "distributional" sense. Note, however, that the actual domains of Q and P are not given by the formula of the preceding theorem. For example, the characteristic function $\chi_{[-1,1]}$ is in the domain of Q , but the function $(q, p) \mapsto q \langle \chi_{[-1,1]} | \Lambda(q, p) \chi_{[-1,1]} \rangle$ is not (Lebesgue)-integrable, as is easily seen by calculating the explicit form of the function.

3. COVARIANT QUANTIZATION

We now take the set Ω of the preceding section to be a locally compact second countable unimodular topological group, henceforth denoted as G , and let $\mathcal{B}(G)$ denote the Borel σ -algebra of G . Fix λ to be a Haar measure in G . Let $\mathcal{T}(\mathcal{H})$ denote the Banach space of trace class operators on the Hilbert space \mathcal{H} , and let $\text{Aut}(\mathcal{T}(\mathcal{H}))$ denote the set of linear, positive,

trace norm preserving bijections from $\mathcal{T}(\mathcal{H})$ onto itself. We consider it equipped with the topology given by the functionals $\text{Aut}(\mathcal{T}(\mathcal{H})) \ni \beta \mapsto \text{Tr}[A\beta(T)] \in \mathbb{C}$, where $A \in L(\mathcal{H})$ and $T \in \mathcal{T}(\mathcal{H})$.

Assume further that there is a continuous group homomorphism $\beta : G \rightarrow \text{Aut}(\mathcal{T}(\mathcal{H}))$ and a constant $d > 0$, satisfying

$$(1) \quad \int \text{Tr}[P_1\beta(g)(P_2)]d\lambda = d \quad \text{for all one-dimensional projections } P_1, P_2 \text{ on } \mathcal{H}.$$

We now consider quantizations connected to the structure of G given by the homomorphism β , in the following sense: A map $\Gamma : \mathcal{F}(G) \rightarrow \mathcal{O}(\mathcal{H})$ with the property that $\Gamma(f) \in L(\mathcal{H})$ for all bounded functions $f \in \mathcal{F}(G)$ is said to be β -covariant, if $\beta(g)^*(\Gamma(f)) = \Gamma(f(g \cdot))$ for all $g \in G$ and all bounded functions $f \in \mathcal{F}(G)$.

If Γ is such that it can be represented by the operator integral with respect to an observable E , (i.e. Γ satisfies the conditions of Theorem 2), then it is straightforward to verify that Γ is β -covariant if and only if the observable E is β -covariant in the following sense: An observable $E : \mathcal{B}(G) \rightarrow L(\mathcal{H})$ is said to be β -covariant if $\beta(g)^*(E(B)) = E(g^{-1}B)$ for all $g \in G$, $B \in \mathcal{B}(G)$.

Covariant observables are essential in quantum mechanics, and hence they have been studied quite extensively. The canonical examples of covariant observables are constructed e.g. in [7], and there are (at least) two completely different ways to obtain their characterization: a direct approach [16, 38, 19], which uses the theory of integration with respect to vector measures, and a group theoretical approach [6]. The most general of these characterizations is in [6]. In our context of a unimodular group, the characterization is given by the following theorem.

Theorem 3. *Let T be a positive operator of trace one. Then there is a β -covariant observable $E^T : \mathcal{B}(G) \rightarrow L(\mathcal{H})$, such that*

$$(2) \quad E^T(B) = d^{-1} \int_B \beta(g)(T)d\lambda(g)$$

in the ultraweak sense for each $B \in \mathcal{B}(G)$. Conversely, assume that $E : \mathcal{B}(G) \rightarrow L(\mathcal{H})$ is a β -covariant observable. Then there is a unique positive operator T of trace one, such that $E = E^T$.

If E is a β -covariant observable, we call the corresponding trace-one positive operator T the *generating operator* for E . Thus, for a β -covariant observable E , we have

$$D(f, E) = \{\varphi \in \mathcal{H} \mid g \mapsto f(g)\langle\psi|\beta(g)(T)\varphi\rangle \text{ is } \lambda\text{-integrable for each } \psi \in \mathcal{H}\}$$

and

$$\langle\psi|L(f, E)\varphi\rangle = d^{-1} \int f(g)\langle\psi|\beta(g)(T)\varphi\rangle d\lambda(g)$$

for all $\varphi \in D(f, E)$ and $\psi \in \mathcal{H}$, where T is the generating operator for E .

According to the Wigner theorem, each $\beta(g)$ has the form $\beta(g)(T) = U(g)TU(g)^*$ for some unitary or antiunitary operator $U(g)$, which is unique up to a phase factor, so that $\beta(g)$ corresponds to the associated equivalence class of unitary operators (see e.g. [5, p. 19] or [16, p. 22]). It follows that, in the case where G is connected, the map $g \mapsto U(g)$ is a weakly (Borel) measurable projective unitary representation of G , where each $U(g)$ is chosen from the equivalence class corresponding to $\beta(g)$ by means of some (measurable) section (see [16, p. 23]

and [5, pp. 30, 100]). The relation (1) gives the so-called square integrability condition

$$\int |\langle \psi | U(g) \varphi \rangle|^2 d\lambda(g) = d$$

for all unit vectors $\psi, \varphi \in \mathcal{H}$. Clearly, for each $\varphi, \psi \in \mathcal{H}$ there is a $g \in G$ such that $\langle \psi | U(g) \varphi \rangle \neq 0$. This implies that the closed linear span of $\{U(g)\varphi \mid g \in G\}$ is dense in \mathcal{H} for each $\varphi \in \mathcal{H}$, which means that the projective representation $g \mapsto U(g)$ is irreducible. The β -covariance condition for an observable E takes the form $U(g)^* E(B) U(g) = E(g^{-1}B)$ for all $g \in G$, $B \in \mathcal{B}(G)$.

Hence, we know that each covariant quantization map $\Gamma : \mathcal{F}(G) \rightarrow \mathcal{O}(\mathcal{H})$, which can be represented by an operator integral, is of the form $\Gamma = \Gamma^T := L(\cdot, E^T)$ for some generating operator T . It is worth noting that in certain cases the observables produced by the quantization scheme associated with a map Γ^T are never spectral measures. Namely, the irreducibility of U implies the following, perhaps well-known result.

Proposition 3. *Assume that G is connected, and that the projective representation U associated with β is strongly continuous. Let $T \in \mathcal{T}(\mathcal{H})$ be positive and of trace one. Then the only projections in the range of E^T are O and I .*

Proof. First we notice that if $E^T(X)$ is a projection for some $X \in \mathcal{B}(G)$ and positive operator T of trace one, then there is a nonzero $\varphi \in \mathcal{H}$, such that $E^{|\varphi\rangle\langle\varphi|}(X)$ is a projection. Indeed, let T be a positive operator of trace one, $\lambda > 0$ an eigenvalue of T (so that $\lambda \leq 1$) and $\varphi \in \mathcal{H}$ an associated eigenvector. Then we can decompose T as $T = \lambda|\varphi\rangle\langle\varphi| + (1 - \lambda)T'$, where $|\varphi\rangle\langle\varphi|$ and T' are positive and of trace one, so we can write $E^T(X) = \lambda E^{|\varphi\rangle\langle\varphi|}(X) + (1 - \lambda)E^{T'}(X)$. Since any projection is an extreme point of the convex set $\{A \in L(\mathcal{H}) \mid 0 \leq A \leq I\}$ (see e.g. [7, p. 19]), it follows that if $E^T(X)$ is a projection, then $E^T(X) = E^{|\varphi\rangle\langle\varphi|}(X)$.

Hence, it suffices to show that for each unit vector $\eta \in \mathcal{H}$, the only projections in the range of $E^{|\eta\rangle\langle\eta|}$ are O and I . Denote $T = |\eta\rangle\langle\eta|$, and assume that there is a projection P in the range of E^T . Then $PE^T(B) = E^T(B)P$ for all $B \in \mathcal{B}(G)$ [27]. Let $\varphi \in \mathcal{H}$. Now

$$\int_B \langle \varphi | P\beta(g)(T)\varphi \rangle d\lambda(g) = \int_B \langle \varphi | \beta(g)(T)P\varphi \rangle d\lambda(g)$$

for all $B \in \mathcal{B}(G)$. Since $g \mapsto \langle \varphi | (P\beta(g)(T) - \beta(g)(T)P)\varphi \rangle$ is continuous, it is thus zero for all $g \in G$. Hence, $P\beta(g)(T) = \beta(g)(T)P$ for all $g \in G$.

Let U_η denote the map $g \mapsto U(g)\eta$. We then have

$$P|U_\eta(g)\rangle\langle U_\eta(g)| = |U_\eta(g)\rangle\langle U_\eta(g)|P$$

for all $g \in G$.

It follows that for each $g \in G$, either $U_\eta(g) \in P(\mathcal{H})$ or $U_\eta(g) \in P(\mathcal{H})^\perp$. Let $f : G \rightarrow \{0, 1\}$ be the function such that $f(g) = 0$ if $U_\eta(g) \in P(\mathcal{H})$ and $f(g) = 1$ if $U_\eta(g) \in P(\mathcal{H})^\perp$. Then f is continuous, when the set $\{0, 1\}$ is equipped with the discrete topology. Indeed, let $g_0 \in G$. Since U_η is continuous, $W = U_\eta^{-1}(\{\varphi \in \mathcal{H} \mid \|U_\eta(g_0) - \varphi\| < \sqrt{2}\})$ is an open set in G containing g_0 . Assume first that $f(g_0) = 0$. Since all vectors $U_\eta(g)$ are of unit length, it follows that $\|U_\eta(g) - U_\eta(g_0)\| = \sqrt{2}$ whenever $f(g) = 1$. Hence, $f(W) \subset \{0\}$. Similarly, if we assume that $f(g_0) = 1$, it follows that $f(W) \subset \{1\}$. This implies that f is continuous. Since G is connected, f cannot be a surjection, so either $U_\eta(g) \in P(\mathcal{H})$ for all $g \in G$, or $U_\eta(g) \in P(\mathcal{H})^\perp$ for all $g \in G$.

But, due to the irreducibility of the projective representation U , the closed linear span of the set $\{U_\eta(g) \mid g \in G\}$ is dense in \mathcal{H} . This is clearly possible only if either $P = I$ or $P = O$. The proof is complete. \square

The following observation is another consequence of the irreducibility of the projective representation associated with β . It uses a calculation similar to that appearing e.g. in [40, p.40] in a different context. Part (a) is mentioned also in [39].

Proposition 4. *Assume that G is connected and let U be the projective representation associated with β . Let $E : \mathcal{B}(G) \rightarrow L(\mathcal{H})$ be a β -covariant observable.*

- (a) *Assume that $f \in \mathcal{F}(G)$ is such that $\tilde{D}(f, E) \subset \tilde{D}(f(g\cdot), E)$ for all $g \in G$. Then $U(g)\tilde{D}(f, E) = \tilde{D}(f, E)$ for all $g \in G$, and either $\tilde{D}(f, E) = \{0\}$ or $\tilde{D}(f, E)$ is dense.*
- (b) *Assume that $f \in \mathcal{F}(G)$ is such that $D(f, E) \subset D(f(g\cdot), E)$ for all $g \in G$. Then $U(g)D(f, E) = D(f, E)$ for all $g \in G$, and either $D(f, E) = \{0\}$ or $D(f, E)$ is dense. Moreover,*

$$(3) \quad U(g)^*L(f, E)U(g) \subset L(f(g\cdot), E)$$

for all $g \in G$.

Proof. Let T be the positive trace one operator associated with E , so that

$$\begin{aligned} \tilde{D}(f, E) &= \{ \varphi \in \mathcal{H} \mid g \mapsto |f(g)|^2 \langle \varphi | U(g)TU(g)^* \varphi \rangle \text{ is } \lambda\text{-integrable} \}; \\ D(f, E) &= \{ \varphi \in \mathcal{H} \mid g \mapsto |f(g)| |\langle \psi | U(g)TU(g)^* \varphi \rangle| \text{ is } \lambda\text{-integrable for all } \psi \in \mathcal{H} \}. \end{aligned}$$

For all $h, g \in G$, we have $U(h)^*U(g) = c(g, h)U(h^{-1}g)$, where $(h, g) \mapsto c(h, g)$ is some torus valued function, so that $U(g)^*U(h) = [U(h)^*U(g)]^* = c(g, h)^{-1}U(h^{-1}g)^*$, and hence

$$(4) \quad U(h)^*U(g)TU(g)^*U(h) = U(h^{-1}g)TU(h^{-1}g)^*.$$

(a) Let $\varphi \in \tilde{D}(f, E)$, and $h \in G$. By the left invariance of the Haar measure and (4), we have

$$\begin{aligned} \int |f(g)|^2 \langle U(h)\varphi | U(g)TU(g)^*U(h)\varphi \rangle d\lambda(g) &= \int |f(g)|^2 \langle \varphi | U(h^{-1}g)TU(h^{-1}g)^*\varphi \rangle d\lambda(g) \\ &= \int |f(hg)|^2 \langle \varphi | U(g)TU(g)^*\varphi \rangle d\lambda(g), \end{aligned}$$

with all the integrands positive. Since $\varphi \in \tilde{D}(f(h\cdot), E)$ by assumption, the last integral is finite, so $U(h)\varphi \in \tilde{D}(f, E)$. Thus $\tilde{D}(f, E)$ is an invariant subspace of the projective representation U , implying that the closure $\overline{\tilde{D}(f, E)}$ is a closed invariant subspace of U . It follows from the irreducibility of U that $\tilde{D}(f, E)$ is either trivial or dense. The fact that $U(h)\tilde{D}(f, E) = \tilde{D}(f, E)$ follows because we have $U(h^{-1}) = c'(h)U(h)^*$ for some torus valued function c' . The proof of (a) is complete.

(b) Let $h \in G$, $\varphi \in D(f, E)$, and $\psi \in \mathcal{H}$. Then by using (4) and the assumption, we get

$$\begin{aligned} \int |f(g)| |\langle \psi | U(g)TU(g)^*U(h)\varphi \rangle| d\lambda(g) &= \int |f(g)| |\langle U(h)U(h)^*\psi | U(g)TU(g)^*U(h)\varphi \rangle| d\lambda(g) \\ &= \int |f(g)| |\langle U(h)^*\psi | U(h^{-1}g)TU(h^{-1}g)^*\varphi \rangle| d\lambda(g) \\ &= \int |f(hg)| |\langle U(h)^*\psi | U(g)TU(g)^*\varphi \rangle| d\lambda(g) < \infty, \end{aligned}$$

so that $U(h)\varphi \in D(f, E)$. Thus $D(f, E)$ is an invariant subspace for U , and hence $D(f, E)$ is either trivial or dense. The fact that $U(h)D(f, E) = D(f, E)$ follows for the same reason as the corresponding one in (a).

Let $h \in G$. By repeating the preceding calculation without the absolute value signs, we get

$$\langle \psi | L(f, E) U(h) \varphi \rangle = \langle U(h)^* \psi | L(f(h \cdot), E) \varphi \rangle = \langle \psi | U(h) L(f(h \cdot), E) \varphi \rangle$$

for each $\psi \in \mathcal{H}$ and $\varphi \in U(h)^* D(f, E) = D(f, E) \subset D(f(h \cdot), E)$, so that (3) holds. \square

Remark.

- (a) If we assume $D(f, E) = D(f(g \cdot), E)$ for all $g \in G$, then Proposition 4 (b) gives the strict operator equality $U(g)^* L(f, E) U(g) = L(f(g \cdot), E)$ for each $g \in G$, with dense domain $D(f, E)$. This resembles the covariance condition for the observable E .
- (b) Since each $E_{\psi, \varphi}$ is a finite measure, it is clear that the conditions of Proposition 4 (a) and (b) are satisfied e.g. by all functions $f \in \mathcal{F}(G)$ with the property that for each $h \in G$ there are nonnegative constants K_h and M_h , such that $|f(hg)| \leq K_h |f(g)| + M_h$ for almost all $g \in G$. In the case where $G = \mathbb{R}^{2n}$ (see the beginning of the next Section), all polynomials of the form $\mathbb{R}^{2n} \ni (x_1, \dots, x_{2n}) \mapsto p(x_i) \in \mathbb{R}$, where $p : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial and $i = 1, \dots, 2n$, are like this.

4. PHASE SPACE QUANTIZATION ON \mathbb{R}^2

Consider the special case where $G = \mathbb{R}^2$, with λ the Lebesgue measure. Fix $\{|n\rangle\}$ to be an orthonormal basis of \mathcal{H} , and let $U : L^2(\mathbb{R}) \rightarrow \mathcal{H}$ be the unitary operator which maps the n th Hermite function h_n to $|n\rangle$. Define $W(q, p) = U W_0(q, p) U^{-1}$, where $(W_0(q, p)f)(t) = e^{i\frac{1}{2}qp} e^{ipt} f(t + q)$, and $\beta : \mathbb{R}^2 \rightarrow \text{Aut}(\mathcal{T}(\mathcal{H}))$ by $\beta(q, p)(T) = W(-q, p) T W(-q, p)^*$. Now β is a continuous group homomorphism, satisfying (1), with $d = 2\pi$, and λ the Lebesgue measure of \mathbb{R}^2 . Let A_{\pm} be the ladder operators associated with the basis $\{|n\rangle\}$, and define Q and P to be the closures of the operators $\frac{1}{\sqrt{2}}(A_+ + A_-)$ and $\frac{1}{\sqrt{2}}i(A_+ - A_-)$, respectively. Then $A_+ = A_-^*$, and Q and P are unitarily equivalent to the position and momentum operators in $L^2(\mathbb{R})$ via U . Let N denote the selfadjoint operator $A_+ A_-$.

According to the general result described above, each positive operator T of trace one generates the map $f \mapsto L(f, E^T)$, where

$$E^T(B) = d^{-1} \int_B \beta(q, p)(T) d\lambda(q, p).$$

The generating operators T of the form $T = \sum_n w_n |n\rangle \langle n|$, where $\sum_n w_n = 1$, and $w_n \geq 0$ for each n , have a special significance, as they are the ones for which E^T is covariant with respect to the phase shifts also, i.e.

$$e^{i\theta N} E^T([0, \infty) \times B) e^{-i\theta N} = E^T([0, \infty) \times (B + \theta))$$

for all $\theta \in [0, 2\pi)$ and $B \in \mathcal{B}([0, 2\pi))$, where $\mathbb{R}^2 = [0, \infty) \times [0, 2\pi)$ and the sum $B + \theta$ is understood modulo 2π . (cf. [23]).

Since $(q, p) \mapsto W(q, p)$ is a strongly continuous projective representation, and \mathbb{R}^2 is connected, Proposition 3 tells us that the range of E^T does not contain nontrivial projections. In particular, the corresponding quantization scheme cannot then produce spectral measures.

In this Section, we inspect the possibility of applying the quantization scheme described earlier to the classical position and momentum variables, using the quantization map $\Gamma^T = L(\cdot, E^T)$ with various generating operators T . For each $k \in \mathbb{N}$, let x^k and y^k denote the functions $(q, p) \mapsto q^k$ and $(q, p) \mapsto p^k$.

The essential question is whether the operator measures $B \mapsto E^T(B \times \mathbb{R})$ and $B \mapsto E^T(\mathbb{R} \times B)$ are uniquely determined by their respective moment operator sets $\{\Gamma^T(x^k) \mid k \in \mathbb{N}\}$ and $\{\Gamma^T(y^k) \mid k \in \mathbb{N}\}$. It is known that this is indeed the case when T is a number state $|n\rangle\langle n|$, $n \in \mathbb{N}$ (see [13]). In that case, the operator sets $\{\Gamma^T(x^k) \mid k \in \mathbb{N}\}$ and $\{\Gamma^T(y^k) \mid k \in \mathbb{N}\}$ are eligible to represent the quantizations of x and y , respectively. As is well known, the associated quantum mechanical observables are unsharp position and momentum observables.

Our goal here is to explicitly determine the operators $\Gamma^T(x^k)$ and $\Gamma^T(y^k)$ for certain generating operators T .

To begin with, we consider the square integrability domains. According to Proposition 4 and the associated Remark, these sets are either dense or trivial. The following two Propositions specify them completely.

Proposition 5. *Let $k \in \mathbb{N}$, let $\eta \in \mathcal{H}$ be a unit vector, and denote $u = U^{-1}\eta \in L^2(\mathbb{R})$.*

- (a) *$\tilde{D}(x^k, E^{|\eta\rangle\langle\eta|}) \neq \{0\}$ if and only if $\eta \in D(Q^k)$, and in this case, $\tilde{D}(x^k, E^{|\eta\rangle\langle\eta|}) = D(Q^k)$.*
- (b) *The statement of (a) holds true, if "x" and "Q" are replaced by "y" and "P".*

Proof. Let $0 \neq \varphi \in \mathcal{H}$ and $f = U^{-1}\varphi \in L^2(\mathbb{R})$. We get

$$\begin{aligned} \int_{\mathbb{R}^2} q^{2k} dE_{\varphi, \varphi}^{|\eta\rangle\langle\eta|}(q, p) &= \frac{1}{2\pi} \int q^{2k} \left(\int |\langle \varphi | W(-q, p) | \eta \rangle|^2 dp \right) dq \\ &= \int q^{2k} \left(\int |F(\overline{u}(\cdot - q)f)(p)|^2 dp \right) dq \\ &= \int q^{2k} \left(\int |u(t - q)|^2 |f(t)|^2 dt \right) dq \\ &= \int \left(\int q^{2k} |u(t - q)|^2 |f(t)|^2 dq \right) dt \\ &= \int \int (t - q)^{2k} |u(q)|^2 |f(t)|^2 dq dt, \end{aligned}$$

where Lemma 2 of [18], the unitarity of the Fourier-Plancherel operator, and Fubini's theorem have been used. (Since all the functions and measures involved are positive, the calculation is valid regardless of whether the integrals are finite or not.)

Now the last integral is finite if and only if φ and η are both in $D(Q^k)$. This is seen as follows.

Assume first that the last integral is finite. Then it follows from Fubini's theorem that $t \mapsto (t - q)^{2k} |f(t)|^2 |u(q)|^2$ is integrable for almost all q , and $q \mapsto (t - q)^{2k} |u(q)|^2 |f(t)|^2$ is integrable for almost all t . Thus $t \mapsto t^{2k} |f(t)|^2$ and $q \mapsto q^{2k} |u(q)|^2$ are integrable. (The fact that $t \mapsto t^{2k} |f(t)|^2$ is integrable is seen as follows: Take $q \in \mathbb{R}$, such that $|u(q)|^2 > 0$ and $t \mapsto (t - q)^{2k} |f(t)|^2 |u(q)|^2$ is integrable. This is possible, since $\|\eta\| > 0$, which implies that $|u(q)|^2 > 0$ in some non-null set. Then use the fact that there exist positive constants A, B, M , such that $At^{2k} \leq (t - q)^{2k} \leq Bt^{2k}$ for $|t| \geq M$. The fact that $q \mapsto q^{2k} |u(q)|^2$ is integrable follows similarly, since we assumed that also $\|\varphi\| > 0$.) Thus f and u are in the domain of the k th power of the position operator in $L^2(\mathbb{R})$, so $\varphi, \eta \in D(Q^k)$.

Conversely, assume that $\varphi, \eta \in D(Q^k)$, so that $t \mapsto t^{2k}|f(t)|^2$ and $q \mapsto q^{2k}|u(q)|^2$ are integrable. Hence also $t \mapsto |t|^l|f(t)|^2$ and $q \mapsto |q|^l|u(q)|^2$ are integrable for all $l \leq 2k$, implying that $(t, q) \mapsto (t - q)^{2k}|u(q)|^2|f(t)|^2$ is integrable over \mathbb{R}^2 . Thus the last integral of the above calculation is finite.

We conclude that $\tilde{D}(x^k, E^\eta) \neq \{0\}$ if and only if $\eta \in D(Q^k)$, and in this case, $\tilde{D}(x^k, E^{|\eta\rangle\langle\eta|}) = D(Q^k)$.

The result concerning $\tilde{D}(y^k, E^{|\eta\rangle})$ is obtained in an analogous manner by using the calculation

$$\begin{aligned} \int_{\mathbb{R}^2} p^{2k} dE_{\varphi, \varphi}^s(q, p) &= \frac{1}{2\pi} \int p^{2k} \left(\int |\langle \varphi | W(-q, p) | \eta \rangle|^2 dq \right) dp \\ &= \int p^{2k} \left(\int |F^{-1}(\overline{Fu}(\cdot - p)Ff)(q)|^2 dq \right) dp \\ &= \int p^{2k} \left(\int |Fu(t - p)|^2 |Ff(t)|^2 dx \right) dp \\ &= \int \left(\int p^{2k} |Fu(t - p)|^2 |Ff(t)|^2 dp \right) dt \\ &= \int \int (t - p)^{2k} |Fu(p)|^2 |Ff(t)|^2 dp dt, \end{aligned}$$

as well as the fact that $P = UF^{-1}U^{-1}QUFU^{-1}$. □

Now we consider the case of an arbitrary positive operator T of trace one. The following elementary fact is needed. The proof is included for the reader's convenience.

Lemma 1. *Let T be a positive operator of trace one. Let (η_n) be an orthonormal sequence and (w_n) a sequence of nonnegative numbers, such that $T = \sum_n w_n |\eta_n\rangle\langle\eta_n|$. Let A be a closed operator. Then*

$$\sum_{n=1}^{\infty} w_n \|A\eta_n\|^2 < \infty,$$

if and only if $A\sqrt{T}$ is a Hilbert-Schmidt operator. (Here we have denoted $\|A\eta_n\| = \infty$ whenever $\eta_n \notin D(A)$, and used the convention $0 \cdot \infty = 0$.) In particular, the convergence of the series is not dependent on the representation of T in terms of (η_n) and (w_n) .

Proof. Let $S = \sum_{n=1}^{\infty} w_n \|A\eta_n\|^2 (\leq \infty)$. Assume first that $S < \infty$, so that, in particular, $\eta_n \in D(A)$ for all those $n \in \mathbb{N}$ for which $w_n > 0$. Let $\varphi \in \mathcal{H}$. Since the series $\sqrt{T} = \sum_n \sqrt{w_n} |\eta_n\rangle\langle\eta_n|$ converges in the operator norm, the vector series $\sum_n \sqrt{w_n} \langle\eta_n|\varphi\rangle \eta_n$ converges to $\sqrt{T}\varphi$ in the norm of \mathcal{H} . Since (η_n) is orthonormal, the Cauchy-Schwartz inequality gives

$$\sum_n \sqrt{w_n} |\langle\eta_n|\varphi\rangle| \|A\eta_n\| \leq \sqrt{S} \|\varphi\| < \infty,$$

so also the series $\sum_n \sqrt{w_n} \langle\eta_n|\varphi\rangle A\eta_n$ converges in norm. Since A is closed, it follows that $\sqrt{T}\varphi \in D(A)$ and $A\sqrt{T}\varphi$ equals the sum of the latter series. In particular, $D(A\sqrt{T}) = \mathcal{H}$. Now the previous inequality shows that $\|A\sqrt{T}\varphi\| \leq \sqrt{S} \|\varphi\|$, so $A\sqrt{T}$ is bounded. Clearly $\sum_{\xi \in \mathcal{K}} \|A\sqrt{T}\xi\|^2 = S < \infty$ if \mathcal{K} is an orthonormal basis of \mathcal{H} which includes all the η_n , so $A\sqrt{T}$ is Hilbert-Schmidt.

Assume then that $A\sqrt{T}$ is a Hilbert-Schmidt operator. Now $\eta_n = w_n^{-\frac{1}{2}}\sqrt{T}\eta_n \in D(A)$ if $w_n > 0$, and $S = \sum_{\xi \in \mathcal{K}} \|A\sqrt{T}\xi\|^2 < \infty$, where \mathcal{K} is an orthonormal basis including all the η_n . \square

Proposition 6. (a) Let $k \in \mathbb{N}$. Then $\tilde{D}(x^k, E^T) \neq \{0\}$ if and only if $Q^k\sqrt{T}$ is a Hilbert-Schmidt operator, and in that case, $\tilde{D}(x^k, E^T) = D(Q^k)$.

(b) The statement in (a) holds true, if "x" and "Q" are replaced by "y" and "P".

Proof. Write T in the form $T = \sum_{n=1}^{\infty} w_n |\eta_n\rangle\langle\eta_n|$, where $\sum_n w_n = 1$, $t_n \geq 0$, and (η_n) is an orthonormal sequence in \mathcal{H} . The series converges in the trace norm, as well as in the operator norm.

For each $\varphi \in \mathcal{H}$, let $A_{\varphi, \varphi}^{\eta}$ be the density function of the positive measure $E_{\varphi, \varphi}^{|\eta\rangle\langle\eta|}$. Since the density function of the measure $E_{\varphi, \varphi}^T$ is $\sum_n w_n A_{\varphi, \varphi}^{\eta_n}$, we have $\varphi \in \tilde{D}(x^k, E^T)$ if and only if the function $x^{2k} \sum_n w_n A_{\varphi, \varphi}^{\eta_n}$ is integrable over \mathbb{R}^2 . In view of the Proposition 5, it is therefore clear that $\tilde{D}(x^k, E^T) \neq \{0\}$ only if $\eta_n \in D(Q^k)$ for all those $n \in \mathbb{N}$ for which $w_n > 0$, and that in any case, $\tilde{D}(x^k, E^T) \subset D(Q^k)$.

Assume now that $\eta_n \in D(Q^k)$ for all $n \in \mathbb{N}$ with $w_n > 0$, and $0 \neq \varphi \in D(Q^k)$. Let $u_n = U^{-1}\eta_n$ for each n . The monotone convergence theorem and the proof of Proposition 5 imply that

$$\int x^{2k} dE_{\varphi, \varphi}^T = \sum_n w_n \int x^{2k} dE_{\varphi, \varphi}^{\eta_n} = \int \int \sum_n (t - q)^{2k} w_n |u_n(q)|^2 |(U^{-1}\varphi)(t)|^2 dt dq$$

(regardless of whether the series converges or not). Now if the above integral is finite (i.e. $\varphi \in \tilde{D}(x^k, E^T)$), then Fubini's theorem gives that $q \mapsto (t - q)^{2k} |(U^{-1}\varphi)(t)|^2 \sum_n w_n |u_n(q)|^2$ is integrable for almost all t , so by the argument similar to that used in the proof of the preceding Proposition, it follows that $\sum_n w_n \|Q^k \eta_n\|^2 = \int \sum_n w_n q^{2k} |u_n(q)|^2 dq < \infty$. On the other hand, if $\sum_n w_n \|Q^k \eta_n\|^2 < \infty$, then each function $q \mapsto \sum_n w_n q^l |u_n(q)|^2$, with $l \leq 2k$ is integrable, so that $(q, t) \mapsto \sum_n (t - q)^{2k} w_n |u_n(q)|^2 |(U^{-1}\varphi)(t)|^2$ is integrable over \mathbb{R}^2 . (Note that since we assumed that $\varphi \in D(Q^k)$, the function $t \mapsto t^l |(U^{-1}\varphi)(t)|^2$ is integrable for each $l \leq 2k$.) Thus $\int x^{2k} dE_{\varphi, \varphi}^T < \infty$, so $\varphi \in \tilde{D}(x^k, E^T)$.

We have proved that $\tilde{D}(x^k, E^T) \neq \{0\}$ if and only if $\eta_n \in D(Q^k)$ for all $n \in \mathbb{N}$ with $w_n > 0$, and

$$\sum_{n=1}^{\infty} w_n \|Q^k \eta_n\|^2 < \infty$$

(where it is understood that $\|Q^k \eta_n\| = \infty$ if $\eta_n \notin D(Q^k)$ and we use the convention $0 \cdot \infty = \infty$). Since Q^k is closed, (a) follows from the preceding Lemma. The statement of (b) is proved similarly, since

$$\int x^{2k} dE_{\varphi, \varphi}^T = \int \int \sum_n w_n (t - p)^{2k} |Fu_n(p)|^2 |FU^{-1}\varphi(t)|^2 dp dt,$$

$P = UF^{-1}U^{-1}QUFU^{-1}$, and P^k is also closed. The proof is complete. \square

Now we proceed to determine the operators $L(x^k, E^T)$ and $L(y^k, E^T)$ for $T = \sum_n w_n |\eta_n\rangle\langle\eta_n|$ satisfying the condition of the preceding Proposition.

Theorem 4. (a) Assume that T satisfies the condition of the previous Proposition (a). Then $L(x^k, E^T) = \sum_{l=0}^k s_{kl}^Q Q^l$, where $s_{kl}^Q = \binom{k}{l} (-1)^{k-l} \text{Tr}[Q^{k-l} T]$, with each $Q^{k-l} T$ a trace class operator.
(b) The statement in (a) holds true, if "(a)", "x" and "Q" are replaced by "(b)", "y" and "P".

Proof. Assume first that $T = |\eta\rangle\langle\eta|$ for $\eta \in D(Q^k)$. Denote $u = U^{-1}\eta \in L^2(\mathbb{R})$. Define a polynomial $p^\eta : \mathbb{R} \rightarrow \mathbb{R}$ by

$$p^\eta(t) = \langle\eta|(t - Q)^k\eta\rangle = \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \langle\eta|Q^{k-l}\eta\rangle t^l.$$

Since p^η is a polynomial of order k , the operator $p^\eta(Q)$ is selfadjoint, and has the domain $D(Q^k)$. Thus by Proposition 5, we have $D(p^\eta(Q)) = D(Q^k) = \tilde{D}(x^k, E^{|\eta\rangle\langle\eta|})$.

Let $\varphi \in \tilde{D}(x^k, E^{|\eta\rangle\langle\eta|}) \subset D(x^k, E^{|\eta\rangle\langle\eta|})$, and $\psi \in \mathcal{H}$. Let $f = U^{-1}\varphi$, $g = U^{-1}\psi$. Since the function

$$(q, p) \mapsto q^k \langle\psi|W(-q, p)|\eta\rangle \overline{\langle\varphi|W(-q, p)|\eta\rangle}$$

is integrable over \mathbb{R}^2 (by the definition of $D(x^k, E^{|\eta\rangle\langle\eta|})$), we get

$$\begin{aligned} \langle\psi|L(x^k, E^{|\eta\rangle\langle\eta|})\varphi\rangle &= \int_{\mathbb{R}^2} q^k dE_{\psi, \varphi}^{|\eta\rangle\langle\eta|}(q, p) \\ &= \frac{1}{2\pi} \int q^k \left(\int \langle\psi|W(-q, p)|\eta\rangle \overline{\langle\varphi|W(-q, p)|\eta\rangle} dp \right) dq \\ &= \int q^k \left(\int \overline{F(\bar{u}(\cdot - q)g)(p)} F(\bar{u}(\cdot - q)f)(p) dp \right) dq \\ &= \int q^k \left(\int \overline{u(t - q)g(t)} u(t - q)f(t) dt \right) dq \\ &= \int \left(\int q^k |u(t - q)|^2 dq \right) \overline{g(t)} f(t) dt \\ &= \int \left(\int (t - q)^k |u(q)|^2 dq \right) \overline{g(t)} f(t) dt, \\ &= \int \langle\eta|(t - Q)^k\eta\rangle \overline{g(t)} f(t) dt \\ &= \langle\psi|p^\eta(Q)\varphi\rangle. \end{aligned}$$

The fifth equality follows from Fubini's theorem, since $(q, t) \mapsto q^k |u(t - q)|^2 \overline{g(t)} f(t)$ is integrable (because of the Cauchy-Schwarz inequality and the square integrability of the maps $(q, t) \mapsto |u(t - q)g(t)|$ and $(q, t) \mapsto |q^k u(t - q)f(t)|$, the latter being a consequence of the proof of Proposition 5.) It follows that $p^\eta(Q) \subset L(x^k, E^{|\eta\rangle\langle\eta|})$.

The equality $p^\eta(Q) = L(x^k, E^{|\eta\rangle\langle\eta|})$ follows from the fact that being selfadjoint, the operator $p^\eta(Q)$ cannot have a proper symmetric extension.

Now we take $T = \sum_n w_n |\eta_n\rangle\langle\eta_n|$ under the condition of the preceding Proposition, so that $\tilde{D}(x^k, E^T) = D(Q^k) = \tilde{D}(x^k, E^{|\eta_n\rangle\langle\eta_n|})$ for each n . Let $\varphi \in D(Q^k)$ and $\psi \in \mathcal{H}$, and f, g be as before.

According to Proposition 1 of [18], we have

$$\begin{aligned}\langle \psi | L(x^k, E^T) \varphi \rangle &= \sum_{n=1}^{\infty} w_n \langle \psi | L(x^k, E^{|\eta_n\rangle\langle\eta_n|}) \varphi \rangle = \sum_{n=1}^{\infty} w_n \langle \psi | p^{\eta_n}(Q) \varphi \rangle \\ &= \sum_{n=1}^{\infty} w_n \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \langle \eta_n | Q^{k-l} \eta_n \rangle \langle \psi | Q^l \varphi \rangle.\end{aligned}$$

Since

$$\|Q^{k-l}\eta_n\|^2 = \int q^{2(k-l)} |(U^{-1}\eta_n)(q)|^2 dq \leq 1 + \int q^{2k} |(U^{-1}\eta_n)(q)|^2 dq = 1 + \|Q^k\eta_n\|,$$

it follows from the preceding Lemma that also each $Q^{k-l}\sqrt{T}$ is defined in all of \mathcal{H} and is a Hilbert-Schmidt operator. Thus each $Q^{k-l}T = Q^{k-l}\sqrt{T}\sqrt{T}$ is defined in all of \mathcal{H} and is a bounded operator of trace class. Thus the series $\sum_n w_n \langle \eta_n | Q^{k-l}\eta_n \rangle$ converges (clearly to $\text{Tr}[Q^{k-l}T]$) for each l , so

$$\langle \psi | L(x^k, E^T) \varphi \rangle = \sum_{l=0}^k s_{kl}^Q \langle \psi | Q^l \varphi \rangle.$$

Since φ was arbitrarily chosen from the set $\tilde{D}(x^k, E^T) = D(Q^k)$ and the operator $\sum_{l=0}^k s_{kl}^Q Q^l$ is selfadjoint, we conclude that $L(x^k, E^T) = \sum_{l=0}^k s_{kl}^Q Q^l$, and the proof of (a) is complete.

The statement (b) is proved similarly by using the unitary equivalence of Q and P . \square

Remark. As mentioned at the beginning of the Section, the uniqueness of the operator measure which gives the moment operators of Theorem 4, is verified only in the case where $T = |n\rangle\langle n|$ for some n . The uniqueness question in the general case remains open.

We close this Section with a remark on another application of our quantization scheme. Consider the function $h(q, p) = \frac{1}{2}(q^2 + p^2)$, i.e. the classical oscillator energy variable. It is known that for each $n \in \mathbb{N}$, the operators $\Gamma^{|n\rangle\langle n|}(h^k)$, $k \in \mathbb{N}$, are the moment operators of the polar margin of the phase space observable $E^{|n\rangle\langle n|}$, and that the marginal observable is uniquely determined by its moments [12]. Thus a quantization of h is given by the set $\{\Gamma^{|n\rangle\langle n|}(h^k) \mid k \in \mathbb{N}\}$ of operators. These operators were determined explicitly in [22]; they are certain polynomials of the usual oscillator Hamiltonian $\frac{1}{2}(Q^2 + P^2)$. The quantized oscillator energy observable is the unsharp number observable (see [4, p. 90]).

5. OPTIMAL PHASE SPACE QUANTIZATION IN \mathbb{R}^2

Consider the situation of the previous Section. At least in the case where $T = |n\rangle\langle n|$, the quantizations of the position and momentum variables x and y corresponding to the covariant quantization map $\Gamma^T = L(\cdot, E^T)$ are the Cartesian margins of E^T , or, equivalently, the sets of the operators $\{\Gamma^T(x^k) \mid k \in \mathbb{N}\}$ and $\{\Gamma^T(y^k) \mid k \in \mathbb{N}\}$. If the margins were projection valued, the quantization of e.g. x would just be the spectral measure of $L(x, E^T)$, with each operator $L(x^k, E^T)$ equal to the corresponding power of $L(x, E^T)$. Although this is not the case, we can still try to find those generating operators T for which the situation would be in some sense close to this ideal situation, where only the first power of x is needed to determine its quantization.

First, we can find the generating operators T which give $L(x, E^T) = Q$ and $L(y, E^T) = P$, so as to make the operators $\Gamma^T(x)$ and $\Gamma^T(y)$ equal to the actual position and momentum operators. In view of Theorem 4, we know that the square integrability domains of $L(x, E^T)$ and $L(y, E^T)$ are nontrivial if and only if $Q\sqrt{T}$ and $P\sqrt{T}$ are Hilbert-Schmidt operators. In that case we have,

$$\begin{aligned} L(x, E^T) &= Q - \text{Tr}[QT]I; \\ L(y, E^T) &= P - \text{Tr}[PT]I. \end{aligned}$$

So if we assume that the square integrability domains of $L(x, E^T)$ and $L(y, E^T)$ are nontrivial, we have $L(x, E^T) = Q$ and $L(y, E^T) = P$ exactly when T is such that $\text{Tr}[QT] = \text{Tr}[PT] = 0$. This occurs, for example, if we choose T to be a mixture of number states, i.e. $T = \sum_n w_n |n\rangle\langle n|$. Then the above Hilbert-Schmidt conditions take the form $\sum_n w_n n < \infty$ [18].

Consider next the operators $\Gamma^T(x^2)$ and $\Gamma^T(y^2)$. According to Theorem 4, they are given by

$$\begin{aligned} L(x^2, E^T) &= Q^2 - 2\text{Tr}[QT]Q + \text{Tr}[Q^2T]I; \\ L(y^2, E^T) &= P^2 - 2\text{Tr}[PT]P + \text{Tr}[P^2T]I, \end{aligned}$$

provided that $Q^2\sqrt{T}$ and $P^2\sqrt{T}$ are Hilbert-Schmidt operators (or, equivalently, that $\tilde{D}(x^2, E^T)$ and $\tilde{D}(y^2, E^T)$ are nontrivial). In order to make the situation close to the spectral measure case, we would like to minimize the "noise" operators $R^T(x) = L(x^2, E^T)^2 - L(x, E^T)^2$ and $R^T(y) = L(y^2, E^T)^2 - L(y, E^T)^2$. Now

$$\begin{aligned} R^T(x) &= (\text{Tr}[Q^2T] - \text{Tr}[QT]^2)I = \text{Var}(Q, T)I; \\ R^T(y) &= (\text{Tr}[P^2T] - \text{Tr}[PT]^2)I = \text{Var}(P, T)I, \end{aligned}$$

on the domains $D(Q^2)$ and $D(P^2)$, respectively, where e.g. $\text{Var}(Q, T)$ denotes the variance of the probability measure $p_T^Q := \text{Tr}[TE^Q(\cdot)]$, with E^Q the spectral measure of Q . The last equalities are obtained as follows: Let $T = \sum_n w_n |\eta_n\rangle\langle \eta_n|$, $p_T^Q = \text{Tr}[TE^Q(\cdot)]$, and $p_n^Q = \langle \eta_n | E^Q(\cdot) | \eta_n \rangle$, where E^Q is the spectral measure of Q . Now $p_T^Q = \sum_n w_n p_n^Q$, with the series converging absolutely in the total variation norm, so we have (by e.g. Lemma 1 of [18]) that

$$\begin{aligned} \text{Tr}[Q^2T] - \text{Tr}[QT]^2 &= \sum_n w_n \langle \eta_n | Q^2 | \eta_n \rangle - \left(\sum_n w_n \langle \eta_n | Q | \eta_n \rangle \right)^2 \\ &= \sum_n w_n \int x^2 dp_n^Q - \left(\sum_n w_n \int x dp_n^Q \right)^2 = \int x^2 dp_T^Q - \left(\int x dp_T^Q \right)^2. \end{aligned}$$

Therefore, $R^T(x) = \text{Var}(Q, T)$. The result $R^T(y) = \text{Var}(P, T)I$ follows similarly. Since $\text{Var}(Q, T)$ and $\text{Var}(P, T)$ are always positive, we see explicitly that $R^T(x)$ and $R^T(y)$ are never zero.

As is well known, the generating operator T can be chosen so that e.g. $R^T(x) = \text{Var}(Q, T)I$ is arbitrarily small (in the sense that $\|R^T(x)\| = \text{Var}(Q, T)$ is such), but then $R^T(y)$ becomes large, because of the inequality $\text{Var}(Q, T)\text{Var}(P, T) \geq \frac{1}{4}$. The product $R^T(x)R^T(y)$ can reach its lower bound $\frac{1}{4}$ only in the case where T is a vector state of minimal uncertainty. If we assume that $\text{Tr}[QT] = 0 = \text{Tr}[PT]$ as discussed before, the operators T that give $R^T(x)R^T(y) = \frac{1}{4}$ are of the form $T = |\eta\rangle\langle \eta|$, with

$$(U^{-1}\eta)(t) = (\sqrt{\pi}\Delta q)^{-\frac{1}{2}} e^{-\frac{t^2}{4(\Delta q)^2}},$$

where $\Delta q > 0$ (in fact, $(\Delta q)^2 = \text{Var}(Q, |\eta\rangle\langle\eta|)$ [35, p. 92]. Moreover, we could require that $R^T(x) = R^T(y)$, so as to make the situation symmetric between x and y . This leaves us with only one generating operator, namely $T = |0\rangle\langle 0|$. Note that this choice indeed gives a quantization of position and momentum, for the associated operator measure is uniquely determined by its moment operators (see Remark of Theorem 4).

6. WHEN IS A POSITIVE OPERATOR MEASURE PROJECTION VALUED?

Let $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ be a positive operator measure. If E is a spectral measure, the first moment $L(x, E)$ is always selfadjoint on the domain $\tilde{D}(x, E)$, and $\int x^2 dE_{\varphi, \varphi} = \|L(x, E)\varphi\|^2$ for all $\varphi \in \tilde{D}(x, E)$. In the case of a general positive operator measure, this need not be true, as the above case of the Cartesian margins of the phase space observable E^T demonstrates. It turns out that this condition is sufficient for a positive operator E to be a spectral measure. The proof of this fact can be found in [1, p. 130]. Since that proof does not contain certain details which are perhaps not obvious, we give a (slightly different) proof here as part (b) of the following Proposition.

An adaptation of the steps leading to the result in [31, p. 466] gives part (a) of the following Proposition. For each $k \in \mathbb{N}$, we let $\tilde{L}(x^k, E)$ denote the restriction of $L(x^k, E)$ to $\tilde{D}(x^k, E)$.

Proposition 7. *Let $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ be a positive operator measure, such that*

$$\int x^2 dE_{\varphi, \varphi} = \|L(x, E)\varphi\|^2$$

for all $\varphi \in \tilde{D}(x, E)$.

(a) *$\tilde{L}(x^n, E) = \tilde{L}(x, E)^n$ for all $n \in \mathbb{N}$.*

(b) *If $\tilde{L}(x, E)$ is assumed to be selfadjoint, then E is projection valued.*

Proof. Let $P : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{K}$ be a Naimark dilation of E into a spectral measure acting on a Hilbert space \mathcal{K} . Let $V : \mathcal{H} \rightarrow \mathcal{K}$ be the associated isometric map, so that $E(B) = V^*P(B)V$ for all $B \in \mathcal{B}(\mathbb{R})$. Denote by $P_{\mathcal{H}}$ the projection VV^* , acting on \mathcal{K} with $V\mathcal{H}$ as its range. (Note that V^*V is the identity operator of \mathcal{H} .) Now $\tilde{L}(x^k, E) = V^*L(x^k, P)V$ for each $k \in \mathbb{N}$ (see [24]). Since P is a spectral measure, we thus have

$$(5) \quad \tilde{L}(x^k, E) = V^*A^kV$$

for all $k \in \mathbb{N}$, where $A = L(x, P)$. Denote $E_1 = \tilde{L}(x, E)$. We prove by induction that for each $n \in \mathbb{N}$,

$$(6) \quad \tilde{D}(x^n, E) = D(E_1^n), \text{ and } A^n V\varphi = V E_1^n \varphi \text{ for all } \varphi \in \tilde{D}(x^n, E).$$

Take first $n = 1$, and let $\varphi \in \tilde{D}(x, E) = D(E_1) = D(AV)$. Since the measures $E_{\varphi, \varphi}$ and $P_{V\varphi, V\varphi}$ are the same, and P is a spectral measure, the assumption implies that

$$\|AV\varphi\|^2 = \int x^2 dP_{V\varphi, V\varphi} = \int x^2 dE_{\varphi, \varphi} = \|E_1\varphi\|^2.$$

Using (5) and the fact that V is isometric, we thus get

$$\|AV\varphi\|^2 = \|E_1\varphi\|^2 = \|V^*AV\varphi\|^2 = \|P_{\mathcal{H}}AV\varphi\|^2.$$

Since $P_{\mathcal{H}}$ is a projection, this means that

$$(7) \quad AV\varphi = P_{\mathcal{H}}AV\varphi = V E_1 \varphi \text{ for all } \varphi \in \tilde{D}(x, E) = D(E_1),$$

i.e. (6) holds for $n = 1$. Now let $k \in \mathbb{N}$, $k > 1$, and assume that (6) holds for $n = k - 1$. Let $\varphi \in \tilde{D}(x^k, E)$. By (5), this implies that $V\varphi \in D(A^k)$, so that $AV\varphi \in D(A^{k-1})$. Since $\varphi \in \tilde{D}(x^k, E) \subset \tilde{D}(x, E)$, it thus follows from (7) that $V(E_1\varphi) = AV\varphi \in D(A^{k-1})$, so (5) and the induction assumption give $E_1\varphi \in \tilde{D}(x^{k-1}, E) = D(E_1^{k-1})$. Hence, $\varphi \in D(E_1^k)$. Conversely, if $\varphi \in D(E_1^k)$, then $\varphi \in \tilde{D}(x, E) = D(AV)$ and $E_1\varphi \in D(E_1^{k-1}) = \tilde{D}(x^{k-1}, E)$, so $AV\varphi = V(E_1\varphi) \in D(A^{k-1})$ by (7) and (5), implying that $V\varphi \in D(A^k)$, i.e. $\varphi \in \tilde{D}(x^k, E)$. Thus, $\tilde{D}(x^k, E) = D(E_1^k)$. Let φ be in this set. Since now $E_1\varphi \in D(E_1^{k-1})$, the induction assumption (along with the fact that $AV\varphi = V(E_1\varphi)$) gives

$$A^k V\varphi = A^{k-1}(AV\varphi) = A^{k-1}V(E_1\varphi) = VE_1^{k-1}(E_1\varphi) = VE_1^k\varphi,$$

completing the induction proof of (6).

Let $n \in \mathbb{N}$. Now (5) and (6) give $\tilde{L}(x^n, E)\varphi = V^*A^nV\varphi = V^*VE_1^n\varphi = E_1^n\varphi$ for all $\varphi \in \tilde{D}(x^n, E) = D(E_1^n)$, so $\tilde{L}(x^n, E) = \tilde{L}(x, E)^n$. This proves (a).

If we assume that $\tilde{L}(x, E)$ is selfadjoint, it follows from (7) that $P_{\mathcal{H}}D(A) \subset D(A)$. This fact is proved in [34], but we include the proof here for the reader's convenience. To that end, let $\psi \in D(A)$, and let $\varphi \in D(E_1)$ be arbitrary. Using (7), we get

$$\langle E_1\varphi | V^*\psi \rangle = \langle VE_1\varphi | \psi \rangle = \langle AV\varphi | \psi \rangle = \langle \varphi | V^*A\psi \rangle,$$

which implies that $V^*\psi \in D(E_1^*)$. Since E_1 is selfadjoint, $V^*\psi \in D(E_1)$, so $P_{\mathcal{H}}\psi = V(V^*\psi) \in VD(E_1)$. But $VD(E_1)$ is contained in $D(A)$, because $D(E_1) = D(AV)$. Thus $P_{\mathcal{H}}\psi \in D(A)$, proving the fact $P_{\mathcal{H}}D(A) \subset D(A)$. In addition, the above calculation shows that $V^*D(A) \subset D(E_1)$, and $E_1V^*\psi = E_1^*(V^*\psi) = V^*A\psi$ for all $\psi \in D(A)$. Combining this with (7), we get

$$P_{\mathcal{H}}A\psi = VV^*A\psi = V(E_1V^*\psi) = VE_1(V^*\psi) = AV(V^*\psi) = AP_{\mathcal{H}}\psi$$

for all $\psi \in D(A)$. Consequently, $P_{\mathcal{H}}A \subset AP_{\mathcal{H}}$. Since A is selfadjoint, this implies that $P_{\mathcal{H}}$ commutes with all the spectral projections $P(B)$ [31, pp. 320, 301]. It follows that each $E(B)$ is a projection [27, Corollary 2.2.2.], so the proof is complete. \square

Remark. As mentioned before, the result appearing in part (b) of the above Proposition can be found in the classic book of Akhiezer and Glazman [1]. The result seems to be somewhat well known (see e.g. [30, 37], both of which refer to the works of Akhiezer and Glazman). However, the fact is given in a much later work [21] without reference to [1] (though we have not been able to convince ourselves of their argumentation), and R. Werner [39, p. 796] only mentions that it holds for normalized compactly supported operator measures. Moreover, Ingarden [17, p. 87] says that all the semispectral measures with the same selfadjoint first moment A , have variances greater than or equal to that of the spectral measure of A . Part (b) of the above Proposition gives more - it asserts that the minimum variance occurs *only* in the case of the spectral measure of A . We note also that the proof given in [17, p. 87] considers only compactly supported semispectral measures, and contains no reference to [1].

Now we get the following characterization for projection valued measures.

Theorem 5. *Let $E : \mathcal{B}(\mathbb{R}) \rightarrow L(\mathcal{H})$ be a positive normalized operator measure, such that $\tilde{L}(x, E)$ is selfadjoint. Then the following conditions are equivalent.*

- (i) E is a spectral measure;
- (ii) $L(x^2, E) = L(x, E)^2$;

(iii) $\int x^2 dE_{\varphi,\varphi} = \|L(x, E)\varphi\|^2$ for all $\varphi \in \tilde{D}(x, E)$.

Proof. Since $\tilde{L}(x, E)$ is selfadjoint, it coincides with its symmetric extension $L(x, E)$.

Assume that (i) holds. Then $\tilde{L}(x^2, E) = \tilde{L}(x, E)^2$ by a standard result of spectral theory. Since E is projection valued, we have also $\tilde{L}(x^2, E) = L(x^2, E)$, so (ii) holds.

Assume (ii). Then we have

$$(8) \quad \int x^2 dE_{\varphi,\varphi} = \|L(x, E)\varphi\|^2$$

for all $\varphi \in D(x^2, E)$. In order to get (iii), we have to establish this identity for vectors in the larger set $D(x, E) = \tilde{D}(x, E)$. Let $\varphi \in \tilde{D}(x, E) = D(x, E)$. Since $L(x, E) = \tilde{L}(x, E)$ is selfadjoint, the closure of the restriction of $L(x, E)$ to the domain of $L(x, E)^2$ is $L(x, E)$ itself [11, p. 1245]. Therefore, by (ii), we can pick a sequence (φ_n) of vectors in $D(x^2, E)$, converging to φ , such that $(L(x, E)\varphi_n)$ converges to $L(x, E)\varphi$. Since $\varphi_n \in D(x^2, E)$ for each n , (8) gives that

$$\lim_n \int x^2 dE_{\varphi_n, \varphi_n} = \lim_n \|L(x, E)\varphi_n\|^2 = \|L(x, E)\varphi\|^2.$$

Since $|E_{\varphi_n, \varphi_n}(B) - E_{\varphi, \varphi}(B)| \leq (\|\varphi_n\| + \|\varphi\|)\|E(\mathbb{R})\|\|\varphi_n - \varphi\|$ for all $n \in \mathbb{N}$ and $B \in \mathcal{B}(\mathbb{R})$, the sequence $(E_{\varphi_n, \varphi_n}(B))$ converges to $E_{\varphi, \varphi}(B)$ uniformly for $B \in \mathcal{B}(\mathbb{R})$, so the sequence $(E_{\varphi_n, \varphi_n})$ of positive measures converges to $E_{\varphi, \varphi}$ in the total variation norm [10, p. 97]. It follows by [22, Lemma A.5] that

$$(9) \quad \int x^2 dE_{\varphi, \varphi} \leq \|L(x, E)\varphi\|^2, \quad \varphi \in D(x, E).$$

It follows e.g. from the proof of Lemma A.2 of [22] (see [3, p. 65]) that $\|L(x, E)\varphi\|^2 \leq \int x^2 dE_{\varphi, \varphi}$ for all $\varphi \in D(x, E)$. Combining this with (9), we get (iii).

Because (iii) implies (i) by the preceding Proposition, the proof is complete. \square

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